APPROXIMATE VALUES OF Π.*

By Wilfred H. Sherk.

Last spring I asked one of my classes to find the approximate value of π after the manner presented in Wentworth's "Plane Geometry." Only one pupil had the perseverence to do the great amount of accurate work necessary to get the result correct to five places of decimals. This fact led me to seek a less laborious way. I tried finding the area of a circle with unit radius. Area equals π. I did this by methods which could be applied in finding the areas of closed plane surfaces other than the circle, with the hope of finding a method so simple and accurate that it could be presented to high school pupils and with the hope of thus furnishing the boys and girls with a mathematical tool which should be of use in subsequent mathematical work. My hope was not realized, yet the various formulæ employed were interesting.

I. Rectangular Method.

The equation, $x^2 + y^2 = 1$, was assumed. The range, $0 - 1$, was divided into one hundred equal parts, and the ordinates corresponding to the abscissas of the points of division were computed. Then,

A. The sum of the series of rectangles inscribed in a quadrant of the circle was computed, and

B. The sum of the series of rectangles circumscribed about a quadrant was obtained.

Four times either sum gave an approximate value of π. Consideration of the definite integral suggested this method.

II. Trapezoidal Rule.

The upper ends of the ordinates computed as above were connected by straight lines thus forming a series of inscribed trapezoids. If $y_0, y_1, y_2, y_3 \cdots y_n$ are the ordinates of the points of division of the range, $0 - 1$, and h represents the length of one of these divisions the areas of the trapezoids are, $\frac{1}{2}h$

* Read before the Rochester Section.

$(y_0 + y_1)$, $\frac{1}{2}h\,(y_1 + y_2)$, $\frac{1}{2}h\,(y_2 + y_3), \cdots$, the sum of which gives

$$\text{Area} = h\,(\tfrac{1}{2}\,y_0 + y_1 + y_2 + y_3 \cdots + y_{n-1} + \tfrac{1}{2}\,y_n).$$

Writing only the coefficients, as is customary in writing such formulæ,

$$\text{Area} = h\,(1/2 + 1 + 1 + 1 + 1 + 1 \cdots + 1 + 1/2).$$

This formula, of course, gave for π a value which was the mean of the results given by A and B of I.

III. Cotes's Method.

The values of π thus far obtained were not correct through two places of decimals. Of course more accurate results could have been obtained by increasing the number of equal parts into which the range, $0 - 1$, was divided, but 100 was as many as my time and patience would allow. It became evident then that if more accurate results were to be obtained, the upper boundary of the little parts into which the quadrant was divided would have to be something other than straight lines. This consideration led to an examination of a number of rules for approximating the area included between two ordinates, a given curve, and the axis of X:

A. Simpson's 1/3 rule, or the parabolic rule.

B. Simpson's 3/8 rule.

C. Weddle's rule.

These and several other rules were found to be special cases of a more general formula attributed to Roger Cotes, a contemporary of Newton and the editor of Newton's "Principia." Cotes made use of Lagrange's well known interpolation formula the derivation of which follows.

Suppose $f\,(x)$ is finite, continuous, and single valued in the range $a - b$, and that it is desired to replace $f\,(x)$ by another function of x that shall differ from $f\,(x)$ little at will. Plot the curve of $y = f(x)$, and divide the range, $a - b$, into n equal parts whose abscissas are $x_0, x_1, x_2, x_3, x_4, \cdots x_n$. Compute the corresponding ordinates, $y_0, y_1, y_2, y_3, y_4, \cdots y_n$. We now have $(n + 1)$ points of $y = f(x)$, given by their coordinates. The problem now is to find a function of x, call it

$F(x)$, whose curve shall pass through these $(n+1)$ points. $F(x)$ has the form,

$$F(x) = S_0 + S_1x + S_2x^2 + S_3x^3 + \cdots S_nx^n,$$

in which the S's are to be determined. One could write $n+1$ equations of the form,

$$y_r = S_0 + S_1x_r + S_2x_r^2 + S_3x_r^3 + \cdots S_nx_r^n,$$

solve for the S's and substitute their values in

$$F(x) = S_0 + S_1x + S_2x^2 + S_3x^3 + \cdots S_nx^n,$$

and thus get $F(x)$.

A procedure which leads to more general results, however, is that followed by Lagrange. He assumed the identity,

$$Q(x) \equiv (x - x_0)(x - x_1)(x - x_2)(x - x_3) \cdots (x - x_n).$$

$F(x)/Qx$ is evidently a proper fraction and can be broken up into partial fractions. To do this make use of a rule thus stated by Byerly, "To find the numerator corresponding to a linear denominator, $X - a$, we have simply to strike that factor out of the denominator of the given fraction, and then substitute a for x in the result." The fraction corresponding to the denominator $x - x_r$ is then

$$\frac{\left.\dfrac{\dfrac{F(x)}{Q(x)}}{x - x_r}\right]_{x=x_r}}{x - x_r}$$

and if the sum of such partial is taken,

$$\frac{F(x)}{Q(x)} = \sum_{r=0}^{r=n}\left\{\frac{\left[\dfrac{\dfrac{F(x)}{Q(x)}}{x - x_r}\right]_{x=x_r}}{x - x_r}\right\} = \sum_{r=0}^{r=n}\left\{y_r\frac{\left[\dfrac{\dfrac{1}{Q(x)}}{x - x_r}\right]_{x=x_r}}{x - x_r}\right\}$$

whence

$$F(x) = \sum_{r=0}^{r=n}\left\{\frac{\dfrac{Q(x)}{x - x_r}}{\left[\dfrac{Q(x)}{x - x_r}\right]_{x=x_r}}\right\}$$

and

$$F(x)=\sum_{r=0}^{r=n}\left\{y_r\frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{r-1})\#(x-x_{r+1})\cdots(x-x_n)}{(x_r-x_0)(x_r-x_1)(x_r-x_2)\cdots(x_r-x_{r-1})\#(x_r-x_{r+1})\cdots(x_r-x_n)}\right\}.$$

Suppose now that the range is from 0 to a

$$x_0=0,\ x_1=\frac{a}{n},\ x_2=\frac{2a}{n},\ x_3=\frac{3a}{n},\ \cdots,\ x_n=\frac{na}{n}$$

whence by substitution and multiplying numerator and denominator by n^n,

$$F(x)=\sum_{r=0}^{r=n}\left\{y_r\frac{nx(nx-a)(nx-2a)\cdots\{nx-(r-1)a\}\#\{nx-(r+1)a\}\cdots(nx-na)}{nx_r(nx_r-a)(nx_r-2a)\cdots\{nx_r-(r-1)a\}\#\{nx_r-(r+1)a\}\cdots(nx_r-na)}\right\}.$$

That is,

$$F(x)=y_0\frac{\#(nx-a)(nx-2a)\cdots(nx-na)}{\#(0-a)(0-2a)\cdots(0-na)}$$

$$+y_1\frac{nx\#(nx-2a)\cdots(nx-na)}{a\#(a-2a)\cdots(a-na)}$$

$$+y_2\frac{nx(nx-a)\#(nx-3a)\cdots(nx-na)}{2a(2a-a)\#(2a-3a)\cdots(2a-na)}$$

$$\cdots\cdots\cdots\cdots$$

$$+y_n\frac{nx(nx-a)(nx-2a)\cdots\{nx-(n-1)a\}\#}{na(na-a)(na-2a)\cdots\{na-(n-1)a\}\#}.$$

(In these formulæ the symbol # indicates the position occupied by a factor which has been cancelled.)

The locus of this equation passes through $(n+1)$ points of the locus of $y=f(x)$ within the range, $0-a$. It is to be noticed that the coefficients of the y's are independent of the given function, $f(x)$, and so can be computed once for all for various values of a and n. Herein lies the beauty and efficiency of Lagrange's interpolation formula.

Now the definite integral, $\int_0^a F(x)dx$, is graphically represented by the area included between the locus of $y = F(x)$, the axes and the ordinate, y_a. Cotes seized upon this fact, and after replacing $f(x)$ by $F(x)$ which differed from $f(x)$ little at will, he found $\int_0^a F(x)dx$. This gave him an area which differed from the area included by $y = f(x)$, the axes, and y_a as little as he chose to make it, the amount of difference depending only upon the patience and accuracy of the computer.

A. Simpson's 1/3, or the Parabolic Rule.

Following Cote's method, let $n = 2$ in Lagrange's formula and get the definite integral,

$$\int_0^a F(x)dx = \frac{a}{6}(y_0 + 4y_1 + y_2).$$

If h represents one of the equal parts into which the range $0 - a$, is divided, then $a = 2h$ and

$$\int_0^a F(x)dx = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now if any range, $0 - c$, is divided into a number of equal parts each equal to a and the formula just derived is applied in succession to each range, a, and the definite integrals summed,

$$\int_0^c F(x)dx = \frac{h}{3}(y_0+4y_1+2y_2+4y_3+2y_4+ \cdots +4y_{m-1}+y_m),$$

or

$$\int_0^c F(x)dx = \frac{h}{3}(1 + 4 + 2 + 4 + 2 + \cdots + 4 + 1).$$

This is Simpson's 1/3 rule, also called the parabolic rule because its derivation is usually made to depend upon the properties of the parabola.

B. Simpson's 3/8 Rule.

If in Lagrange's formula we let $n = 3$ and proceed as in obtaining the parabolic rule,

$$\int_0^c F(x)dx = \frac{3h}{8}(y_0+3y_1+3y_2+2y_3+3y_4+3y_5+ \cdots +3y_{m-1}+y_m),$$

or

$$\int_0^c F(x)dx = \frac{3h}{8}(1 + 3 + 3 + 2 + 3 + 3 + \cdots + 3 + 1).$$

This formula is known as Simpson's 3/8 rule.

C. Weddle's Rule.

If we put n equal to 6 and proceed as above,

$$\int_0^a F(x)dx = \frac{3h}{10}\left[\tfrac{41}{42}y_0 + \tfrac{216}{42}y_1 + \tfrac{27}{42}y_2 + \tfrac{272}{42}y_3 + \tfrac{27}{42}y_4 + \tfrac{216}{42}y_5 + \tfrac{41}{42}y_6\right].$$

Mr. Thomas Weddle called this

$$\int_0^a F(x)dx = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6),$$

or

$$\int_0^a F(x)dx = \frac{3h}{10}(1 + 5 + 1 + 6 + 1 + 5 + 1),$$

or in general

$$\int_0^c F(x)dx = \frac{3h}{10}(1 + 5 + 1 + 6 + 1 + 5 + 2 + 5 + 1 + 6 + 1 + 5 + \cdots + 5 + 1).$$

Cotes carried these formulæ out to the case in which n equals ten. The results, however, are too cumbersome to be useful.

The value of π obtained by these various methods were disappointing. They were as follows:

By circumscribing rectangles	3.1586
By inscribing rectangles	3.1186
By Trapezoidal rule	3.1386
By Parabolic rule	3.1547
By Simpson's 3/8 rule	3.1385
By Weddle's rule (error)	

Of course there are many other ways of approximating the value of π. Perhaps as simple and rapid a way as any is by the method of series taught in differential calculus. However after making myself weary computing the value of π, I took a crude

circular disk, an ordinary scale, and a piece of paper, measured the circumference and the diameter of the disk, divided one by the other, and obtained a result that was surprisingly accurate. I concluded then that until I had more light I would not require my pupils in elementary geometry to compute the value of π. They must show that the ratio of the circumference of a circle to its diameter is constant, but the value of that ratio they may measure.

LAFAYETTE HIGH SCHOOL,
BUFFALO, N. Y.

Fundamentally, the trouble with our system of public education is that children learn a little about a great many things, without gaining much really definite knowledge of anything which is likely to stand them in good stead in later life; and, what is equally bad, they do not acquire methods of accurate thinking. This criticism applies to education in every branch, and has been true from the beginning because our theories have been wrong. But just now it is peculiarly true that our public school system has defects which are so palpable, and for which remedies are so easily found, that we should no longer permit things to remain as they are.

It cannot be too strongly impressed upon the American people that the so-called "three R's" are not grounded in the youthful minds of this generation as they should be. Reading, Writing, and Arithmetic are the tools of the human mind. Without them almost nothing of an intellectual sort can be accomplished. We ought to put the very best tools in the hands of our children, and they should be kept in the best possible condition. Professor Barrett Wendell, of Harvard University, recently told in a lecture how one of his brightest students did not know the letters of the alphabet in their order, and in consequence was much hampered in the use of a dictionary. When it comes to writing, we have retrograded. At various times much stress has been laid on various systems of penmanship which have been in turn adopted and discarded, with the result that few children write so well as they should, or so well as did their forebears at the same age. As to composition, most children are befogged at the simplest test.—JOSEPH M. ROGERS in January *Lippincott's*.